Black Hole Thermodynamics

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1 Introduction and Background

As a black hole gobbles up nearby stars, dust, and unsuspecting experimentalists, its mass, size, spin, and charge can all change. Their rates of change are all tied together:

$$\mathrm{d}M = \frac{\kappa}{8\pi}\mathrm{d}A + \Omega\,\mathrm{d}J + \Phi\,\mathrm{d}Q. \tag{1.1}$$

Here, M is the mass of the black hole, κ is its surface gravity at the event horizon, A is the surface area of the event horizon, Ω is its angular velocity, J is its angular momentum, Φ is its electrostatic potential, and Q is its electric charge. Black holes pull so strongly that no mass can ever escape their clutches; that is, their surface area can never decrease: $\mathrm{d}A \geq 0$.

Physicists noticed a strong analogy between these equations and the laws of thermodynamics. Famously, we now interpret M as the black hole's total internal energy U, the quantity $S \equiv \frac{A}{4}$ as the *Bekenstein-Hawking entropy* of the black hole, and $T \equiv \frac{\kappa}{2\pi}$ as its *Hawking temperature*. Thus black holes obey the second law of thermodynamics ($\mathrm{d}S \geq 0$), and the equation above looks a lot like the first law of thermodynamics, with Ω and Φ interpreted as chemical potentials for the particle-number-like quantities J and Q. For now, set $\mathrm{d}J = \mathrm{d}Q = 0$, so that we're looking at non-spinning, uncharged black holes. Then,

$$\mathrm{d}M = \frac{\kappa}{2\pi}\,\mathrm{d}\left(\frac{A}{4}\right) = T\,\mathrm{d}S = \mathrm{d}U, \qquad \mathrm{d}A \geq 0 \iff \mathrm{d}S \geq 0. \tag{1.2}$$

More recently, it was noticed that the formula above doesn't look quite right: it's missing a work term! One possible proposal is to interpret the *cosmological constant* Λ, which determines the background curvature of spacetime, as a gravitational analog of pressure: $P \equiv -\frac{\Lambda}{8\pi}$. (The prefactor $-\frac{1}{8\pi}$ is conventional.) We use the black hole's actual volume as the volume of the thermodynamic system it defines: $V \equiv V$. (Warning: throughout this problem, some SI units may not make sense. This is because we are using so-called "natural units," where certain constants are set to exactly one: $c = G = \hbar = k_{\mathrm{B}} = 1$.)

2 The Problem

1. The cosmological constant Λ is *constant*, meaning that $\mathrm{d}\Lambda = 0$. With this in mind, which one of $P\,\mathrm{d}V$ or $V\,\mathrm{d}P$ is zero? The one that's zero can be freely added to $T\,\mathrm{d}S$ in the first law of thermodynamics in equation (1.2). Do so, keeping everything in terms of the thermodynamic variables P, V, S, T, U.

2. The resulting expression should be incorrect! What's wrong with it?

3. To fix things up, we should re-interpret the black hole mass M as one of the three other thermodynamic potentials. Which one should M be so that our results make sense?

4. Notice that the volume $V = V$ and the entropy $S = \frac{A}{4}$ are not independent any more for black holes. Assuming that the black hole's event horizon is a sphere, find the relation between them, expressing V in terms of S or vice versa. This provides more evidence that $U = U(S, V)$ is not really the right thermodynamic potential to use.

5. We're about to build a heat engine out of a black hole. But first, prove a key result: adiabats and isochores must be the same for black holes. (!!!)

6. Use the fact that $Q = T\Delta S$ along isotherms, together with the results of previous parts, to compute the efficiency of a black-hole Carnot engine, and confirm that you get the Carnot efficiency. Marvel at how much quicker this calculation is than the typical derivation of the Carnot efficiency, and notice that you've also inadvertently also computed the efficiency of the Stirling cycle.

7. Take a look at sections 1 and 2 of this paper: `https://arxiv.org/pdf/1404.5982.pdf`. (If you know general relativity, read further!) Thermodynamics is alive and well!

8. In 1974, Stephen Hawking showed that the Hawking temperature is given by $T = \frac{1}{8\pi M}$. Determine κ in terms of M, and conclude that small black holes are dangerous.

9. Compute the heat capacity (at constant pressure) of a black hole: what do you find, and what does it imply about throwing matter into a black hole?

10. Hawking also showed that black holes *radiate*, losing mass in the process. Conclude once again that small black holes are more dangerous than large black holes (hint: what happens to their temperature?).

11. Notice that the existence of Hawking radiation causes M, and hence A, to decrease, in apparent contradiction to the second law of thermodynamics! Suggest a resolution to this problem, which begets what is known as the *black hole information paradox*. Then give your academic institution of choice a call, and collect instant tenure and fame.

3 Solutions: Black Hole Heat Engine

The cosmological constant Λ is constant, meaning that $\mathrm{d}\Lambda = 0$. With this in mind, which one of $P\,\mathrm{d}V$ or $V\,\mathrm{d}P$ is zero? The one that's zero can be freely added to $T\,\mathrm{d}S$ in the first law of thermodynamics in equation (1.2). Do so, keeping everything in terms of the thermodynamic variables P, V, S, T, U. The resulting expression should be incorrect! What's wrong with it?

Solution. Since $\mathrm{d}\Lambda = 0$, we also have

$$P = -\frac{\Lambda}{8\pi} \implies \mathrm{d}P = -\frac{\mathrm{d}\Lambda}{8\pi} = 0 \implies V\,\mathrm{d}P = 0. \tag{3.1}$$

We are free to add zero to any expression whatsoever, so let's add it to $\mathrm{d}M$ in (1.2) above:

$$\mathrm{d}M = T\,\mathrm{d}S = T\,\mathrm{d}S + 0 = T\,\mathrm{d}S + V\,\mathrm{d}P = \mathrm{d}U \qquad \text{(wrong!)} \tag{3.2}$$

What just happened? Well, in a desperate attempt to save the claim "$\mathrm{d}U = T\,\mathrm{d}S$" from being wrong, we added the work-like term $V\,\mathrm{d}P$ (which, happily, is always zero) in the hopes that this would salvage the thermodynamic identity. Alas, the true thermodynamic identity is $\mathrm{d}U = T\,\mathrm{d}S - P\,\mathrm{d}V$, and this doesn't match our result above.

To fix things up, we should re-interpret the black hole mass M as one of the three other thermodynamic potentials. Which one should M be so that our results make sense?

Solution. Enthalpy! Recall that the correct thermodynamic identity for enthalpy is

$$\mathrm{d}H = T\,\mathrm{d}S + V\,\mathrm{d}P. \tag{3.3}$$

Matching this to our result above, we find it suggestive to identify $\mathrm{d}M$ with $\mathrm{d}H$, and therefore (because enthalpy is a function of state) M with H.

Notice that the volume $V = V$ and the entropy $S = \frac{A}{4}$ are not independent any more for black holes. Assuming that the black hole's event horizon is a sphere, find the relation between them, expressing V in terms of S or vice versa. This provides more evidence that $U = U(S, V)$ is not really the right thermodynamic potential to use.

Solution. Recall that the volume of the black hole interior and its entropy are given by

$$V = \frac{4}{3}\pi R^3, \qquad S = \frac{A}{4} = \pi R^2, \tag{3.4}$$

where R is the Schwarzschild radius of the black hole. While it's true that $\mathrm{d}V/\mathrm{d}R = A$, we seek a purely algebraic relation between S and V that doesn't involve R at all. To that end, we can do the following manipulation:

$$V^{2/3} = \left(\frac{4\pi}{3}\right)^{2/3} R^2 \implies S = \pi R^2 = \left(\frac{3}{4\pi}\right)^{2/3} \pi V^{2/3}. \tag{3.5}$$

We're about to build a heat engine out of a black hole. But first, prove a key result: adiabats and isochores must be the same for black holes. (!!!)

Solution. This sounds complicated, but it's not: adiabats are curves where $\mathrm{d}S = 0$, while isochores are curves where $\mathrm{d}V = 0$. The relation we discovered above shows that whenever $\mathrm{d}S = 0$, we must also have $\mathrm{d}V = 0$, and vice versa. Thinking physically, if the volume inside a spherical black hole is constant, its radius and therefore its area must also be constant.

Use the fact that $Q = T\Delta S$ along isotherms, together with the results of previous parts, to compute the efficiency of a black-hole Carnot engine, and confirm that you get the Carnot efficiency. Marvel at how much quicker this calculation is than the typical derivation of the Carnot efficiency, and notice that you've also inadvertently also computed the efficiency of the Stirling cycle. Take a look at sections 1 and 2 of this paper: `https://arxiv.org/pdf/1404.5982.pdf`. (If you know general relativity, read further!) Thermodynamics is alive and well!

Solution. We are in an excellent position to read sections 1 and 2 of the following paper: `https://arxiv.org/pdf/1404.5982.pdf`. Section 2 does the efficiency calculation and also reveals that the Stirling cycle (isotherms and isochores) is the same as the Carnot cycle (isotherms and adiabats) for heat engines that use black holes as a working substance.

4 Solutions: Heat Capacity and the Second Law

In 1974, Stephen Hawking showed that the Hawking temperature is given by $T = \frac{1}{8\pi M}$. Determine κ in terms of M, and conclude that small black holes are dangerous.

Solution. Let's relate the surface gravity, temperature, and mass of the black hole:

$$T = \frac{1}{8\pi M} = \frac{\kappa}{2\pi} \implies \kappa = \frac{1}{4M}. \tag{4.1}$$

The smaller M is, the smaller its radius will be; the calculation above shows that the gravity at the "surface" of the black hole will be stronger the smaller the black hole is. Every black hole contains a *singularity* where the curvature of spacetime becomes infinite; smaller black holes pull less strongly, but their event horizons shrink around the singularity faster than their gravity falls off. Standing at the event horizon of a smaller black hole, you'll experience stronger spacetime curvature, and therefore stronger gravity, than if you'd been dropped off at the event horizon of Sgr A*, the supermassive black hole at the center of our galaxy.

Compute the heat capacity (at constant pressure) of a black hole: what do you find, and what does it imply about throwing matter into a black hole?

Solution. The heat capacity at constant pressure is given by

$$C_P = \left(\frac{\partial H}{\partial T}\right)_P = \left(\frac{\partial M}{\partial T}\right)_\Lambda, \tag{4.2}$$

so it behooves us to find M as a function of T. This is straightforward:

$$T = \frac{1}{8\pi M} \implies M = \frac{1}{8\pi T} \implies C_P = \frac{\partial}{\partial T}\left(\frac{1}{8\pi T}\right)_\Lambda = -\frac{1}{8\pi T^2}. \tag{4.3}$$

Evidently black holes have negative heat capacity! This is absolutely bizarre: they get *cooler* when energy is added to the system, and they *heat up* if energy taken away. By the second law, $\mathrm{d}A \geq 0$ prevents this from ever happening, so normally we'd just expect black holes to get cooler and cooler as the universe progresses. However...

Hawking also showed that black holes radiate, losing mass in the process. Conclude once again that small black holes are more dangerous than large black holes (hint: what happens to their temperature?).

Solution. If a black hole radiates, it loses mass, and by $T = 1/(8\pi M)$, its temperature rises. It continues to radiate as a perfect blackbody, its temperature rising without bound, until finally its temperature becomes infinite as it evaporates, boiling itself away and releasing a huge burst of energy in its last moments. On one hand, we definitely don't want to be near a small black hole as this happens. On the other hand, this is a gross violation of $\mathrm{d}A \geq 0$.

There has been much hand-wringing about what to do about this, and the real story of the information paradox lies rather deeper. One "quick fix," proposed by Hawking, was to modify the definition of S to include the entropy of the radiation let out by the black hole: $S \equiv S_{\mathrm{BH}} + S_{\mathrm{rad}}$. It might be hoped that as $\mathrm{d}S_{\mathrm{BH}} = \frac{1}{4}\mathrm{d}A$ falls, $\mathrm{d}S_{\mathrm{rad}}$ rises to compensate. Unfortunately, Hawking's paper showed that $S_{\mathrm{rad}} \equiv 0$: Hawking radiation carries no information at all, and has no entropy. This is really where the information paradox comes from: it is the "generalized entropy" $S_{\mathrm{BH}} + S_{\mathrm{rad}}$ that seems to violate thermodynamics, and many physicists have since been looking for a fault in Hawking's calculation.

Notice that the existence of Hawking radiation causes M, and hence A, to decrease, in apparent contradiction to the second law of thermodynamics! Suggest a resolution to this problem, which begets what is known as the black hole information paradox. Then give your academic institution of choice a call, and collect instant tenure and fame.

"Solution." It is now believed that $S_{\mathrm{rad}} \neq 0$, and that Hawking radiation somehow encodes information about the black hole interior as the black hole evaporates. People suspected that this entropy should come from what are called nonperturbative effects in quantum gravity, but just two years ago, a milestone paper was published which figured out how to reconstruct the black hole interior using a tool called the Euclidean gravitational path integral and (sort of) without resorting to nonperturbative physics. Thus the information paradox has been "solved" using tools that many had thought were too simplistic or coarse to know about the full content of the paradox-free theory. There are many caveats: only certain models have been studied in detail, and it is not yet clear what lessons these new results have to teach us, nor is the story they tell fully understood.